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A UNIFIED APPROACH TO FRACTIONAL CALCULUS PERTAINING TO I-FUNCTIONS

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ABSTRACT

In this paper we study a pair of unified and extended fractional integral operators involving the multivariable H-Function, I-Function and general class of polynomials. During the course of our study, we establish five theorems pertaining to Mellin transforms of these operators. Further, some properties of these operators have also been investigated. On account of the general nature of the functions involved herein, a large number of (known and new) fractional integral operators involving simpler functions can be obtained. For the sake of illustration, some special cases of our main result have been recorded here.

KEYWORDS: Multivariable H – Function, I – Function, General Class of Polynomials, Fractional Integral, Mellin Transform

1. INTRODUCTION

We recall here the following definitions required for the present study:

The general class of polynomials introduced and studied by Srivastava [17] is defined as

$$S_V^U[x]=\sum_{\ell=0}^{[V/U]}\frac{(-V)_{U\ell}A_{V,\ell}}{\ell!}x^{\ell} \tag{1.1}$$

where U, V are arbitrary positive integers and the coefficient $A_{V,\ell}(V,\ell\geq 0)$ are arbitrary constants, real as complex.

The H-function of several complex variables, introduced and studied by Srivastava and Panda [20] is defined and represented in the following form:

$$H\begin{bmatrix}\gamma_1\nu\\ \vdots\\ \gamma_n\nu\end{bmatrix}=H^{0,\lambda(u',\nu');\ldots;[u^{(r)},\nu^{(r)}]}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}\times\left[\begin{matrix}Z_1\\ \vdots\\ Z_r\end{matrix}\middle|\begin{matrix}[(a):(\theta';\ldots;\theta^r]:[(b'):\phi';\ldots;(b^{(r)}):\phi^{(r)}\\ [(c):(\psi';\ldots;\psi^{(r)}]:[(d'):\delta'];\ldots;(d^{(r)}):\delta^{(r)}\end{matrix}\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}U_1(s_1)\cdots U_r(s_r)V(s_1,\ldots,s_r)\,z_1^{s_1}\cdots z_r^{s_r}\,ds_1\cdots ds_r, \tag{1.2}$$

$$U_i(s_i)=\frac{\prod_{j=1}^{u^{(i)}}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{\nu^{(i)}}\Gamma(1-b_j^{(i)}+\phi_j^{(i)}s_1)}{\prod_{j=u^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=\nu^{(i)}+1}^{\nu^{(i)}}\Gamma(b_j^{(i)}+\phi_j^{(i)}s_1)},\ \forall\, i\in\{1,\ldots r\} \tag{1.3}$$

$$V(s_1,\ldots s_r)=\frac{\prod_{j=1}^{\lambda}\Gamma\left(1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}s_1\right)}{\prod_{j=\lambda+1}^{A}\Gamma\left(a_j\sum_{i=1}^{r}\theta_j^{(i)}s_1\right)\prod_{j=1}^{C}\Gamma\left(1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}s_1\right)} \tag{1.4}$$

and $\omega=\sqrt{-1}$

For the conditions of existence on the several parameters of the H-function of several complex variables, we refer the reader to H. M. Srivastava et al. [19, p. 251-253, Eqns. (c.2) to (c.8)].

The I-function is defined and represented as [16] :

$$I^{m,n}_{p_{i'},q_{i'}:r'}[z]=I^{m,n}_{p_{i'},q_{i'}:r'}\left[z\,\middle|\,\begin{matrix}(e_{j'},E_{j'})_{1,n};(e_{j'i'},E_{j'i'})_{n+1,p_{i'}}\\(f_{j'},F_{j'})_{1,m};(f_{j'i'},F_{j'i'})_{m+1,q_{i'}}\end{matrix}\right]$$

$$=\frac{1}{2\pi\omega}\int_L\phi(\xi)\,z^{\xi}\quad d\xi \tag{1.5}$$

where

$$\phi(\xi)=\frac{\prod_{j'=1}^{m}\Gamma(f_{j'}-F_{j'}\xi)\prod_{j'=1}^{n}\Gamma(1-e_{j'}+E_{j'}\xi)}{\sum_{i'=1}^{r'}\left\{\prod_{j'=m+1}^{q_{i'}}\Gamma(1-f_{j'i'}-F_{j'i'}\xi)\prod_{j'=n+1}^{p_{i'}}\Gamma(e_{j'i'}-E_{j'i'}\xi)\right\}} \tag{1.6}$$

and $\omega=\sqrt{-1}$. For the conditions on the several parameters of the I-function, one can refer to [16].

The Mellin transform of f(x) will be denoted by M[f(x)] or by F(s). If p and y are real, we write $s = p^{-1} + iy$. If $p \geq 1$, $f(x) \in Lp\,(0, \infty)$, then for

$$p=1,\ M[f(x)]=F(s)=\int_0^{\infty}x^{s-1}f(x)dx, \tag{1.7}$$

and

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)x^{-s}\,ds, \tag{1.8}$$

under suitable conditions on the variables and the parameters.

For $p > 1$,

$$M[f(x)]=F(s)=\ell.i.m.\int_{1/x}^{x}f(x)x^{s-1}dx, \tag{1.9}$$

where ℓ.i.m. denotes the usual limit in the mean for L_p-spaces.

The pair of new extended fractional integral operators studied here is defined as

$$Q_{\gamma_n}^{\alpha,\beta}[f(x)]=tx^{-\alpha-t\beta-1}\int_0^x y^{\alpha}(x^t-y^t)\times H\begin{bmatrix}\gamma_1\nu\\ \vdots\\ \gamma_n\nu\end{bmatrix}$$

$$\times\prod_{j=1}^{k} I^{m_j,n_j}_{p_{i'j},q_{i'j};r'}\left[z_j\left(\frac{y^t}{x^t}\right)^{a'_j}\left(1-\frac{y^t}{x^t}\right)^{b'_j}\left|\begin{matrix}(e_{j'j},E_{j'j})_{1,n_j};(e_{j'i'j},E_{j'i'j})\\(f_{j'j},F_{j'j})_{1,m_j};(f_{j'i'j},F_{j'i'j})_{m_j+1,q_{i'j}}\end{matrix}\right.\right]$$

$$\times\prod_{i=1}^{r} S_{V_i}^{U_i}\left[z_i\left(\frac{y^t}{x^t}\right)^{g_i}\left(1-\frac{y^t}{x^t}\right)^{h_i}\right]\psi\left(\frac{y^t}{x^t}\right)f(y)dy \tag{1.10}$$

and

$$R_{\gamma_n}^{\rho,\beta}[f(x)]=tx^{\rho}\int_x^{\infty} y^{-\rho-t\beta-1}(y^t-x^t)^{\beta}\times H\begin{bmatrix}\gamma_1\mu\\ \vdots\\ \gamma_1\mu\end{bmatrix}$$

$$\times\prod_{j=1}^{k} I^{m_j,n_j}_{p_{i'j},q_{i'j};r'}\left[z_j\left(\frac{x^t}{y^t}\right)^{a'_j}\left(1-\frac{x^t}{y^t}\right)^{b'_j}\left|\begin{matrix}(e_{j'j},E_{j'j})_{1,n_j};(e_{j'i'j},E_{j'i'j})\\(f_{j'j},F_{j'j})_{1,m_j};(f_{j'i'j},F_{j'i'j})_{m_j+1,q_{i'j}}\end{matrix}\right.\right]$$

$$\times\prod_{i=1}^{r} S_{V_i}^{U_i}\left[z_i\left(\frac{x^t}{y^t}\right)^{g_i}\left(1-\frac{x^t}{y^t}\right)^{h_i}\right]\psi\left(\frac{x^t}{y^t}\right)f(y)dy, \tag{1.11}$$

where $\nu=\left(\frac{y^t}{x^t}\right)^{u_i}\left(1-\frac{y^t}{x^t}\right)^{v_i}$, $\mu=\left(\frac{x^t}{y^t}\right)^{u_i}\left(1-\frac{x^t}{y^t}\right)^{v_i}$ and t, u_i and v_i are positive numbers. The kernels $\psi\left(\frac{y^t}{x^t}\right)$ and $\psi\left(\frac{x^t}{y^t}\right)$ appearing in (1.10) and (1.11) respectively, are assumed to be continuous functions such that the integrals make sense for wide classes of functions f(x).

The conditions for the existence of these operators are as follows :

(I) $f(x)\in Lp\,(0,\infty)$,

(II) $1\le p,\ q<\infty,\ p^{-1}+q^{-1}=1$,

(III) $\mathrm{Re}(\alpha+ta'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}u^{(i)}\frac{c^{(i)}_{j''}}{\psi^{(i)}_{j''}})>-q^{-1}$,

(IV) $\quad \mathrm{Re}\left(\beta+tb'_j \frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n} v^{(i)} \frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}\right)>-q^{-1},$

(IV) $\quad \mathrm{Re}\left(\rho+ta'_j \frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n} v^{(i)} \frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}\right)>-p^{-1},$

where $j''=1, 2, \ldots, u^{(n)}$; $i' = 1, 2, \ldots r'$.

Condition (I) ensures that both operators defined (1.10) and (1.11) belong to Lp $(0,\infty)$.

These operators are extensions of fractional integral operators defined and studied by several authors like Erdelyi [3], Kober [7], Love [9], Saigo et al. [10], Saxena and Kumbhat [13,14,15] , Goyal et al. [5], Saxena and Kiryakova [12], etc.

2. MAIN THEOREMS

Theorem 2.1. If $f(x) \in L_p(0, \infty)$, $1 \le p \le 2$, [or $f(x) \in M_p(0, \infty)$, $p>2$]

$$p^{-1}+q^{-1}=1, \mathrm{Re}\left[\alpha+ta'_j \frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n} u^{(i)} \frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}>-q^{-1}\right],\ \mathrm{Re}\left[\beta+tb'_j \frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n} v^{(i)} \frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}>-q^{-1}\right],$$

for $j''=1, 2, \ldots, u^{(n)}$ and the integrals present are absolutely convergent, then

$$M\ \{Q_{\gamma_n}^{\alpha,\beta}[f(x)]\}=M\{f(x)\}R_{\lambda_n}^{\alpha-s+1,\beta}[1]. \tag{2.1}$$

where M_p $(0, \infty)$ stands for the class of all functions $f(x)$ of $L_p(0, \infty)$ with $p>2$, which are inverse Mellin-transforms of the functions of $L_p(-\infty,\infty)$.

Proof. On taking Mellin transform of (1.10), we have

$$M\ \{Q_{\lambda_n}^{\alpha,\beta}[f(x)]\} = \int_0^\infty x^{s-1}\int_0^x \left\{tx^{-\alpha-t\beta-1}\int_0^x y^{\alpha}(x^t-y^t)^{\beta} \times H\begin{bmatrix}\gamma_1 \nu\\ \vdots \\ \gamma_n \nu\end{bmatrix}\right.$$

$$\times \prod_{j=1}^{k} I_{p_{i'j},q_{i'j}:r'}^{m_j,n_j}\left[z_j\left(\frac{y^t}{x^t}\right)^{a'_j}\left(1-\frac{y^t}{x^t}\right)^{b'_j}\middle|\begin{matrix}(e_{j'j},E_{j'j})_{1,n_j};(e_{j'i'j},E_{j'i'j})\\ (f_{j'j},F_{j'j})_{1,m_j};(f_{j'i'j},F_{j'i'j})_{m_j+1,q_{i'j}}\end{matrix}\right]$$

$$\left.\times\prod_{i=1}^{r} S_{V_i}^{U_i}\left[z_i\left(\frac{y^t}{x^t}\right)^{g_i}\left(1-\frac{y^t}{x^t}\right)^{h_i}\right]\psi\left(\frac{y^t}{x^t}\right)f(y)dy\right\}dx\,. \tag{2.2}$$

Now, after changing the order of integration, which is permissible under the conditions stated, the result (2.1) follows easily in view of (1.11).

Theorem 2.2. If $f(x) \in L_p(0, \infty)$, $1 \le p \le 2$, [or $f(x) \in M_p(0, \infty)$, $p>2$]

$$p^{-1}+q^{-1}=1, \mathrm{Re}\left[\beta+tb'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c^{(i)}_{j''}}{\psi^{(i)}_{j''}}\right]>-q^{-1},$$

$$\mathrm{Re}\left[\rho+ta'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c^{(i)}_{j''}}{\psi^{(i)}_{j''}}\right]>-q^{-1}$$

for $j''=1, 2, \ldots, u^{(n)}$ and the integrals present are absolutely convergent, then

$$M\{R^{\alpha,\beta}_{\gamma_n}[f(x)]\}=M\{f(x)\}Q^{\rho+s-1,\beta}_{\gamma_n}[1]. \tag{2.3}$$

Proof. On taking Mellin transform of (1.11), we have

$$M\{R^{\rho,\beta}_{\gamma_n}[f(x)]\}=\int_0^\infty x^{s-1}\Bigg\{tx^\rho\int_x^\infty y^{-\rho-t\beta-1}(y^t-x^t)^\beta \times H\begin{bmatrix}\gamma_1\mu\\ \vdots\\ \gamma_n\mu\end{bmatrix}$$

$$\times\prod_{j=1}^{k} I^{m_j,n_j}_{p_{i'j},q_{i'j};r'}\left[z_j\left(\frac{x^t}{y^t}\right)^{a'_j}\left(1-\frac{x^t}{y^t}\right)^{b'_j}\middle|\begin{matrix}(e_{j'j},E_{j'j})_{1,n_j};(e_{j'i'j},E_{j'i'j})\\(f_{j'j},F_{j'j})_{1,m_j};(f_{j'i'j},F_{j'i'j})_{m_j+1,q_{i'j}}\end{matrix}\right]$$

$$\times\prod_{i=1}^{r} S^{U_i}_{V_i}\left[z_i\left(\frac{x^t}{y^t}\right)^{g_i}\left(1-\frac{x^t}{y^t}\right)^{h_i}\right]\psi\left(\frac{x^t}{y^t}\right)f(y)dy\Bigg\}dx\,.$$

Now, after changing the order of integration, the result (2.3) can be easily obtained in view of (1.10)

Theorem 2.3. If $f(x)\in L_p(0,\infty)$, $p^{-1}+q^{-1}=1$, $v(x)\in L_p(0,\infty)$,

$$\mathrm{Re}\left[\alpha+ta'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}u^{(i)}\frac{c^{(i)}_{j''}}{\psi^{(i)}_{j''}}\right]>-q^{-1},$$

$$\mathrm{Re}\left[(\beta+tb'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c^{(i)}_{j''}}{\psi^{(i)}_{j''}}\right]>-q^{-1}$$

for $j''=1, 2, \ldots, u^{(n)}$ and the integrals present are absolutely convergent, then

$$\int_0^\infty v(x)Q^{\alpha,\beta}_{\gamma_n}[f(x)]dx=\int_0^\infty f(x)R^{\alpha,\beta}_{\gamma_n}[v(x)]dx\,. \tag{2.4}$$

Proof. The result (2.4) can be easily established in view of (1.10) and (1.11).

3. INVERSION FORMULAE

Theorem 3.1 If $f(x)\in L_p(0,\infty)$, $1\le p\le 2$, [or $f(x)\in M_p(0,\infty)$, p□>□2],

$$p^{-1}+q^{-1}=1, \mathrm{Re}\Big(\alpha+ta'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}u^{(i)}\frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}\Big)>-q^{-1}, \mathrm{Re}\left[\beta+tb'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}\right]>-q^{-1},$$

for $j''=1, 2, \ldots, u^{(n)}$ and the integrals present are absolutely convergent and

$$Q_{\gamma_n}^{\alpha,\beta}[f(x)=[v(x)], \tag{3.1}$$

then

$$f(x)=\int_0^\infty y^{-1}[v(y)]\left[h\left(\frac{x}{y}\right)\right]dy, \tag{3.2}$$

where

$$h(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}y^{-1}\frac{x^{-s}}{R(s)}ds, \tag{3.3}$$

and

$$R(x)=R_{\gamma_n}^{\alpha-s+1,\beta}\ [1]. \tag{3.4}$$

Proof. On taking Mellin transform of (3.1) and then applying Theorem 2.1, we get

$$M\{f(x)\}=\frac{M\{v(x)\}}{R(s)}$$

which on inverting leads to

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}\frac{M\{v(x)\}}{R(s)}ds$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}}{R(s)}\left\{\int_0^\infty y^{s-1}[v(y)]dy\right\}ds.$$

Further, on changing the order of integration, we have

$$f(x)=\int_0^\infty y^{1}[v(y)]\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{y}\right)^{-s}\frac{1}{R(s)}ds\right\}dy.$$

Now, in view of (3.3), we easily arrive at (3.2).

Theorem 3.2. If $f(x) \in L_p(0, \infty)$, $1 \le p \le 2$, [or $f(x) \in M_p(0, \infty)$, $p>2$],

$$p^{-1}+q^{-1}=1, \mathrm{Re}\left[\beta+tb'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}v^{(i)}\frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}\right]>-q^{-1},\ \mathrm{Re}\left[\rho+ta'_j\frac{f_{j'i'j}}{F_{j'i'j}}+t\sum_{i=1}^{n}u^{(i)}\frac{c_{j''}^{(i)}}{\psi_{j''}^{(i)}}\right]>-p^{-1},$$

for $j''=1, 2, \ldots, u^{(n)}$ and the integrals present are absolutely convergent, and

$$R_{\gamma_n}^{\rho,\beta}[f(x)=[w(x)], \tag{3.5}$$

then

$$f(x) = \int_0^\infty y^{-1}[w(y)]\left[G\left(\frac{x}{y}\right)\right]dy, \tag{3.6}$$

where

$$G(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}}{T(s)}ds, \tag{3.7}$$

and

$$T(x)=Q_{\gamma_n}^{\rho+s-1,\beta}\ [1]. \tag{3.8}$$

Proof. On taking Mellin transform of (3.5) and then applying Theorem 2.2, we get

$$M\{f(x)\}=\frac{M\{w(x)\}}{T(s)}$$

which on inverting leads to

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\,\frac{M\{w(x)\}}{T(s)}ds = \frac{1}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{x^{-s}}{T(s)}\left\{\int_0^\infty y^{s-1}[w(y)]dy\right\}\ ds$$

Further, on changing the order of integration, we have

$$f(x) = \int_0^\infty y^{-1}[w(y)]\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{x}{y}\right)^{-s}\frac{1}{T(s)}\,ds\right\}dy\,.$$

Now, (3.6) follows directly in view of (3.5).

4. GENERAL PROPERTIES

The properties given below are immediate consequences of the definitions (1.10) and (1.11)

$$x^{-1}Q_{\gamma_n}^{\alpha,\beta}\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right] = R_{\gamma_n}^{\alpha,\beta}[f(x)], \tag{4.1}$$

$$x^{-1}R_{\gamma_n}^{\rho,\beta}\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right] = Q_{\gamma_n}^{\rho,\beta}[f(x)], \tag{4.2}$$

$$x^{u}Q_{\gamma_n}^{\alpha,\beta}[f(x)] = Q_{\gamma_n}^{\alpha-u,\beta}[x^{u}f(x)], \tag{4.3}$$

$$x^{u}R_{\gamma_n}^{\rho,\beta}[f(x)] = R_{\gamma_n}^{\rho+u,\beta}[x^{u}f(x)]. \tag{4.4}$$

The properties given below express the homogeneity of the operators Q and R respectively.

If $I_{\gamma_n}^{\alpha,\beta}[f(x)] = [\nu(x)]$ then

$$Q_{\gamma_n}^{\alpha,\beta}[f(cx)] = [\nu(cx)] \tag{4.5}$$

If $R_{\gamma_n}^{\rho,\beta}[f(x)] = [Z(x)],$ then

$$R_{\gamma_n}^{\rho,\beta}[f(cx)] = [Z(cx)]. \tag{4.6}$$

5. SPECIAL CASES

(i) In (1.10) and (1.11) if we set r = 1 = $A_{0,0}$, $V_1 = 0$, the general class of polynomials reduces to unity and also taking k = 1 = r_1, the I-function reduces to Fox's H-function [4] and further specifying the parameters appropriately in view of the relationship[6],

$$\frac{t^{rq-\nu-1}}{\Gamma(r)} H_{1,2}^{1,1}\left[-at^q \left| \begin{matrix} (1-r,1 \\ (0,1), 1+\nu\text{-}rq, q \end{matrix} \right. \right] = G_{q,\nu,r}[a,t],$$

we obtain the following pair of fractional integral operators in terms of Lorenzo-Hartely G-function [8] :

$$Q_{\gamma_n,e}^{\alpha,\beta}[f(x)] = t\Gamma(e)^{-\alpha-t\{\beta-(\frac{\alpha'}{b'}+1)(eb'-\nu'-1)\}-1} \times \int_0^x y^{-\alpha-t\frac{\alpha'}{b'}(eb'-\nu'-1)} (x^t - y^t)^{\beta-eb'+\nu'+1}$$

$$\times H\begin{bmatrix} \gamma_1\nu \\ \vdots \\ \gamma_n\nu \end{bmatrix} G_{b',\nu',e}\left[z, \left(\frac{y^t}{x^t}\right)^{\frac{\alpha'}{b'}} \left(1-\frac{y^t}{x^t}\right) \right] \psi\left(\frac{y^t}{x^t}\right) f(y)dy \tag{5.1}$$

and

$$R_{\gamma_n,e}^{\rho,\beta}[f(x)] = t\Gamma(e) x^{\rho-\frac{t\alpha'}{b'}(eb'-\nu'-1)} \int_0^\infty y^{-\rho-t\{\beta-(\frac{\alpha'}{b'}+1)(eb'-\nu'-1)\}-1} (y^t - x^t)^{\beta-eb'+\nu'+1}$$

$$\times H\begin{bmatrix} \gamma_1\mu \\ \vdots \\ \gamma_n\mu \end{bmatrix} G_{b',\nu',e}\left[z, \left(\frac{x^t}{y^t}\right)^{\frac{\alpha'}{b'}} \left(1-\frac{x^t}{y^t}\right) \right] \psi\left(\frac{x^t}{y^t}\right) f(y)dy. \tag{5.2}$$

(ii) Further, taking e=1 in (5.1) and (5.2), in view of the relationship [6]

$$t^{q-\nu-1} H_{1,2}^{1,1}\left[-at^\nu \left| \begin{matrix} (0,1) \\ (0,1), 1+\nu - q, q \end{matrix} \right. \right] = R_{q,\nu}[a,t],$$

we get the following pair of fractional integral operators containing Lorentzo-Hartley R-function [8] :

$$Q^{\alpha,\beta}_{\gamma_n,1}[f(x)] = tx^{-\alpha-t\{\beta-(\frac{\alpha'}{b'}+1)(eb'-v'-1)\}-1} \int_0^x y^{-\alpha-t\frac{\alpha'}{b'}(b'-v'-1)} (x^t - y^t)^{\beta-b'+v'+1}$$

$$\times H\begin{bmatrix}\gamma_1 v\\ \vdots\\ \gamma_n v\end{bmatrix} R_{b',v',e}\left[z,\left(\frac{y^t}{x^t}\right)^{\frac{\alpha'}{b'}}\left(1-\frac{y^t}{x^t}\right)\right]\psi\left(\frac{y^t}{x^t}\right)f(y)dy \tag{5.3}$$

and

$$R^{\rho,\beta}_{\gamma_n,1}[f(x)]=tx^{\rho-\frac{t\alpha'}{b'}(b'-v'-1)}\times\int_0^\infty y^{-\rho-t\{\beta-(\frac{\alpha'}{b'}+1)(b'-v'-1)\}-1}(y^t-x^t)^{\beta-b'+v'+1}$$

$$\times H\begin{bmatrix}\gamma_1 \mu\\ \vdots\\ \gamma_n \mu\end{bmatrix} R_{b',v'}\left[z,\left(\frac{x^t}{y^t}\right)^{\frac{\alpha'}{b'}}\left(1-\frac{x^t}{y^t}\right)\right]\psi\left(\frac{x^t}{y^t}\right)f(y)dy\,. \tag{5.4}$$

(iii) On setting λ = A, $u^{(i)}$ =1, $v^{(i)}$ = $B^{(i)}$, $D^{(i)} \to D^{(i)}$ + 1, where i = 1, …, n in (1.1) and (1.2), the H-function of several complex variables transforms to the generalized Lauricella function of several complex variables [18] and, we get the following fractional integral operators :

$$\Im^{\alpha,\beta}_{\gamma_n,}[f(x)] = Btx^{-\alpha-t\beta-1}\times\int_0^x y^{\alpha}(x^t-y^t)^{\beta}$$

$$F^{A:B';\dots;B^{(r)}}_{C:D';\dots;D^{(r)}}\left[\begin{matrix}[1-(a):\theta',\dots\theta^{(r)}]:[1\text{-}(b'):\phi'];\dots;[1-(b^{(r)}):\phi^{(r)}];\\ [1-(c):\psi',\dots,\psi^{(r)}]:[1\text{-}(d'):\delta'];\dots;[1\text{-}(d^{(r)}:\delta^{(r)}];\end{matrix};-\gamma_1 v,\dots,-\gamma_n v\right]$$

$$\times\prod_{j=1}^{k} I^{M'_j,N'_j}_{M''_{ij},N''_{ij}:r'}\left[z_j\left(\frac{y^t}{x^t}\right)^{a'_j}\left(1-\frac{y^t}{x^t}\right)^{b'_j}\middle|\begin{matrix}(e_{M''_{ij}},E_{M''_{ij}})\\(f_{N''_{ij}},F_{N''_{ij}})\end{matrix}\right]$$

$$\times\prod_{i=1}^{r} S^{m_i}_{n_i}\left[z_i\left(\frac{y^t}{x^t}\right)^{g_i}\left(1-\frac{y^t}{x^t}\right)^{h_i}\right]\psi\left(\frac{y^t}{x^t}\right)f(y)dy \tag{5.5}$$

and

$$\Re^{\rho,\beta}_{\gamma_n,}[f(x)]=Btx^{\rho}\int_x^\infty y^{-\rho-t\beta-1}(y^t-x^t)^{\beta}$$

$$\times F^{A:B';\dots;B^{(r)}}_{C:D';\dots;D^{(r)}}\left[\begin{matrix}[1-(a):\theta',\dots\theta^{(r)}]:[1\text{-}(b'):\phi'];\dots;[1-(b^{(r)}):\phi^{(r)}];\\ [1-(c):\psi',\dots,\psi^{(r)}]:[1\text{-}(d'):\delta'];\dots;[1\text{-}(d^{(r)}:\delta^{(r)}];\end{matrix};-\gamma_1\mu,\dots,-\gamma_n\mu\right]$$

$$\times\prod_{j=1}^{k} I^{M'_j,N'_j}_{M''_{ij},N''_{ij}:r'}\left[z_j\left(\frac{x^t}{y^t}\right)^{a'_j}\left(1-\frac{x^t}{y^t}\right)^{b'_j}\middle|\begin{matrix}(e_{M''_{ij}},E_{M''_{ij}})\\(f_{N''_{ij}},F_{N''_{ij}})\end{matrix}\right]$$

$$\times \prod_{i=1}^{r} S_{n_i}^{m_i}\left[z_i\left(\frac{x^t}{y^t}\right)^{g_i}\left(1-\frac{x^t}{y^t}\right)^{h_i}\right]\psi\left(\frac{x^t}{y^t}\right)f(y)dy. \tag{5.6}$$

where

$$B=\frac{\prod_{j=1}^{A}\Gamma(1-a_j)\prod_{j=1}^{B'}\Gamma(1-b'_j)\cdots\prod_{j=1}^{B^{(r)}}\Gamma(1-b_j^{(r)})}{\prod_{j=1}^{C}\Gamma(1-c_j)\prod_{j=1}^{D'}\Gamma(1-d'_j)\cdots\prod_{j=1}^{D^{(r)}}\Gamma(1-d_j^{(r)})} \tag{5.7}$$

The operators earlier defined by Chaurasia and Srivastava [2], Saxena et al., [11], pp. 212-213, Banerji and Sethi [1], can also be easily derived by assigning suitable values to the parameters occurring in (1.1) and (1.2).

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